Using meaningful contexts to promote understanding of

Pronumerals

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Introduction

Developing a conceptual understanding of elementary algebra has been the focus of a number of recent articles in this journal. Baroudi (2006) advocated problem solving to assist students' transition from arithmetic to algebra, and Shield (2008) described the use of meaningful contexts for developing the concept of function. Samson (2011, 2012) also made use of contexts in order to promote ideas of generalisation and equivalent expressions, while Green (2008, 2009) described the use of spreadsheets for investigating functions and solving equations in meaningful contexts. Although many authors promote the use of meaningful contexts there has been little evidence of any positive effect of such approaches. This article describes approaches to teaching algebra in two recent independent projects, one in Australia and one in New Zealand, both of which made extensive use of meaningful contexts. The three aspects of pronumerals (generalised numbers, variables and unknowns) were taught using real contexts to associate meaning with the pronumeral involved. Both projects demonstrated a positive impact of the approaches on junior secondary students' understandings of pronumerals. These findings suggest that classroom teachers should explore the use of meaningful contexts for teaching algebra.

Variables, generalised numbers, and unknowns

Confusion between different meanings of letter symbols is likely to be a source of difficulty for students (MacGregor & Stacey, 1997). The terms 'pronumerals' and 'variables' are often used interchangeably to refer to letters representing numbers, but there are a number of different meanings that students need to understand. 'Pro-numeral,' (literally 'for a number') is the term we shall use in this article to describe a letter symbol used in elementary algebra. A pronumeral may represent a generalised number, a variable, a specific unknown or a parameter.

- Generalised numbers are used in identities such as $a + b = b + a$, which describe that something is true for any number, and also for structured situations, such as one more than any number n is $n + 1$. The salient point about a generalised number is that it could be any number and the statement about it is always true.
- Treating a pronumeral as a variable in a functional relationship such as $y = x + 1$ is closely related to a generalised number in a structured situation. However, not only could the variable take any value (providing it is allowed within the domain of the function) but it does take different values so that another variable changes in response and hence specifies the function.
- A third use of pronumerals is as specific unknowns, as in $2n + 1 = 7$. There is only one value of n which makes this equation true and so the value can be found.
- Finally, pronumerals may be used as parameters, as in $an + b = c$. (but that is beyond the comprehension of most students studying elementary algebra!).

Algebra and the curriculum

There are many contemporary approaches to the teaching of algebra that emphasise the development of understanding rather than just procedural skills. These include:
- generalisation, in which the emphasis is on generalised numbers and exploring structured situations,
- problem solving, in which the emphasis is on unknowns and using equations as a problem-solving strategy,
- functional approaches, in which the emphasis is on variables and using equations to describe patterns,
- modelling, in which the emphasis is on using equations to describe real world situations.

The use of meaningful contexts, in which algebra is used to describe real-world situations, is integral to all these approaches. The approaches should not be viewed as being mutually exclusive because aspects of each one should always be included in any other. However, whichever approach is taken it is important to make explicit the different meanings of pronumerals.

The *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority, 2012) promotes teaching with understanding through its four Proficiency Strands, understanding (making connections across concepts and representations), fluency (choosing appropriate methods and carrying them out efficiently), problem solving (modelling and investigating unfamiliar or meaningful situations), and reasoning (explaining and justifying strategies and conclusions) (Mulligan, Cavanagh, & Keanan-Brown, 2012). Similarly, the New Zealand Curriculum (Ministry of Education, 2007) specifically states that all mathematics should be taught in a range of meaningful contexts and that the achievement objectives should be addressed by students solving problems and modelling situations. The curricula in both countries are therefore consistent with teaching algebra with understanding.

The Multifaceted Variable Approach project (Australia)

In the Australian Multifaceted Variable Approach project researchers from Macquarie University worked with teachers at an independent girls' school in metropolitan Sydney, studying Year 7 students (age 12–13 years). The students were in four classes which had been streamed according to mathematical ability level at the start of Year 7. Class 1 (high ability) and Class 3 (medium ability) formed a comparison group, while Class 2 (medium ability) and Class 4 (low ability) formed the experimental group. A professional development workshop for the teachers of the experimental classes was conducted at the start of each year. The workshops introduced teachers to the multifaceted variable approach, which involved teaching the three aspects of pronumeral (unknowns, generalised numbers and variables), making use of a wide variety of real-life contexts. The teachers of the comparison classes were not provided with professional development and followed a more conventional programme that focused on patterns for generalisation, evaluation and simplification of linear expressions, and solving linear equations.

The Teaching Algebra Conceptually project (New Zealand)

The New Zealand Teaching Algebra Conceptually project was based on a conception of algebraic thinking as awareness of mathematical structure, rather than as a collection of rules and procedures to be learnt. Researchers from the University of Otago worked in partnership with five teachers of Year 9 classes (age 13–14 years) from two secondary schools. The goal was to develop approaches to teaching algebra that enhanced understanding by students. The project focused on diagnostic assessment of the algebraic knowledge and strategies of the students in order to design learning experiences that addressed identified needs. Rather than pursuing any one particular teaching approach the team collaborated to develop approaches the teachers believed best met the needs of their own students. The teachers then captured their lessons on video and shared selected excerpts and work samples with the others in the project. There were regular fortnightly meetings of the team and four one-day workshops during the year. The experiences of the teachers were used to refine the teaching approaches for the other teachers to try.

Teaching approaches

In Australia the multifaceted variable approach used with the experimental group was based on *Working Mathematically: Activities that Teach Patterns and Algebra* (McMaster & Mitchelmore, 2009). The students studied algebra during Terms 3 and 4 in two teaching blocks of two to three weeks' duration each. The three aspects of pronumerals (generalised numbers, variables and unknowns) were taught simultaneously using real contexts to associate meaning with the pronumeral involved. Multiple representations (tables, graphs, natural language and algebraic expressions) were used for a large variety of rich contexts. An example of a context is shown in Figure 1.

Different starting points

One day Remy's younger sister, Gina, came to the football stadium to help her carry a rolled up banner to the top of the stairs. One girl held each end of the roll. Gina started walking up the stairs when Remy was on stair number 4. When the banner starts its way up the stairs, Remy is on stair number 4. Gina is at the bottom of the stairs (i.e., on stair number 0). Together they walk up the stairs, climbing one stair with each stride.

Figure 1. Context for using multiple representations (McMaster & Mitchelmore, 2009, p. 30)

This functional approach was designed to highlight the role of variables in linear relationships, stimulate discussions about mathematical situations, provide opportunities for developing mathematical language, develop problem-solving strategies, and to explore mathematical relationships by making generalisations and justifying conclusions. This contrasted with the conventional programme taught to the comparison group, in which students briefly studied patterns and then moved quickly to substitution, simplification of algebraic expressions and solving linear equations. The conventional programme therefore devoted considerably more time to the practice of basic algebraic skills often considered necessary for further learning of algebra.

In New Zealand, although the teaching approaches were responsive to the particular needs of each class of students, there was a great deal in common between teachers and a consensus was achieved on effective ways to help students learn algebra. A full description of the approach can be found at https://blogs.otago.ac.nz/tac. Considerable value was placed on a comprehensive diagnostic assessment, which described in detail the students' algebraic knowledge and strategies rather than providing a score. Consistent with the findings of Warren (2003), the diagnostic assessment revealed that many students did not have a good understanding of arithmetic structure, inverse operations or equivalence. Students' understandings of pronumerals were also documented, as well as their strategies for solving equations, expressing generality and finding relationships between variables. This assessment information was used to design teaching activities for whole classes and for working with individuals. The phrase "algebra everywhere" was used to describe how the teachers integrated algebra into the whole mathematics programme, rather than teaching it as an isolated topic.

For example, when teaching equivalent fractions such as $\frac{3}{4}=\frac{9}{12}$ and $\frac{55}{25}=\frac{11}{5}$ by multiplying or dividing numerators and denominators by the same factor, this was generalised to $\frac{ac}{ab}=\frac{c}{b}$, with the generalisation and algebraic notation made explicit to the students.

A variety of rich contexts including sports results, geometric patterns, costs for school events, many very similar to those used in the Australian project, were used throughout the year to make the algebra meaningful. Furthermore, because the team had observed that students' very informal written working and lack of correct mathematical vocabulary was an impediment to learning, care was taken to model correct use of mathematical vocabulary, unfamiliar terms were defined, algebraic notation and conventions were explicitly taught when they were needed, and correct setting out was modelled in all board work. When algebraic skills were identified, they

were described as tools for students to put in their 'toolboxes'. This metaphor was used to promote acceptance and understanding of the skills, and when students were solving problems in any context they were encouraged to select and use algebraic tools purposefully. This approach avoided skills being taught in isolation, as students frequently used their tools.

Impact of the Australian project on student achievement

The Australian project used post-tests to evaluate the impact of the teaching approach. Despite the lower overall mathematical ability of the experimental classes (Classes 2 and 4) compared to the comparison classes (Classes 1 and 3), the experimental classes were at least as successful as the corresponding comparison classes on all measures except solving linear equations. It needs to be noted, however, that the experimental group had not been taught to solve equations, while the comparison group had spent considerable time on this aspect. The lowest ability experimental class (Class 4), in particular, were considerably more successful than the comparison class (Class 3) in writing algebraic expressions to model given situations and describing a tabular relationship in words.

Students' errors in the comparison and experimental groups were just as revealing as the proportions who answered correctly or incorrectly because the various types of errors revealed misconceptions. The main student errors observed were considering the letter (pronumeral) as an object or label, and conjoining errors. As shown in Figure 2, students in the experimental classes made far fewer errors of these kinds than students in the comparison classes.

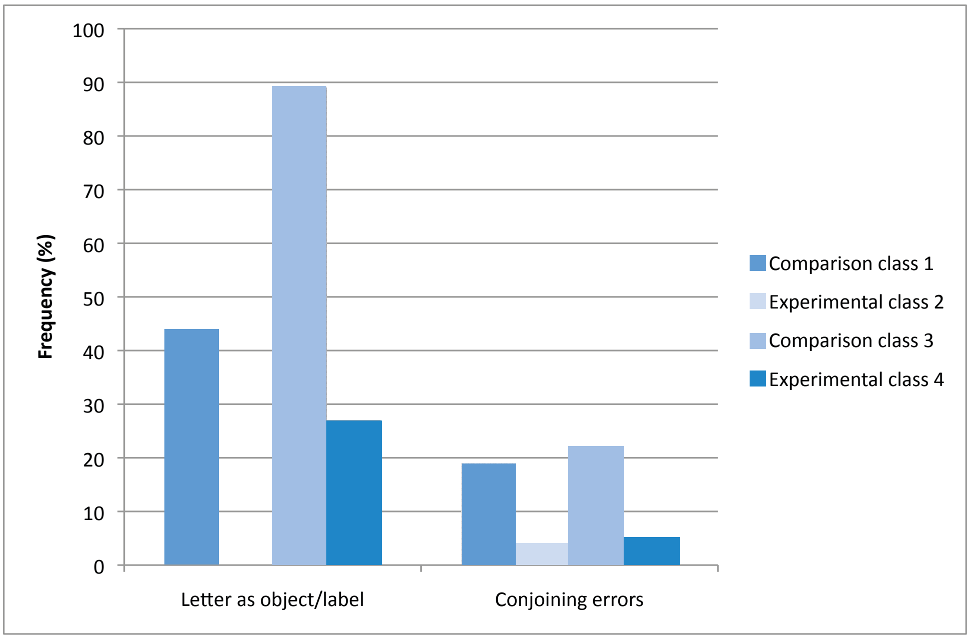

Figure 2. Student errors in post-test.

These errors are exemplified in students' answers to the following question:

> Sarah's mother gave her 2 times as many chocolates as Hannah.
> a) If Hannah has x chocolates, then Sarah will have … chocolates.
> b) When her father came home, he gave each of the girls 5 more chocolates. Describe the number of chocolates each girl has using x. Show your working.

Two common answers given by students to represent Sarah's chocolates in part (a) of the question were: xx chocolates, and 2 chocolates. Students who gave the answer xx appear to be treating the pronumeral x as if it was itself a chocolate and may have been lining up the chocolates side by side to represent Sarah's two chocolates. Those students who chose 2 as their answer may well have been thinking of x as representing one chocolate because x is the same as a chocolate and twice one is two, or alternatively may have arbitrarily decided that Hannah initially had one chocolate. Other students appear to have treated the pronumeral x as a label, using x simply to indicate chocolates, and giving answers to part (b) of $x + 5x = 6x$, $2x + 5x = 7x$ (rather like treating $2a + 3b$ as representing 2 apples and 3 bananas). Examples of conjoining errors (incorrectly bringing two or more terms together into one term) were $x + 5 = 5x$ and $2x + 5 = 7x$. The tendency to conjoin terms incorrectly is likely to occur because students do not fully understand what a pronumeral represents and prefer a single term solution.

Impact of the New Zealand project on student achievement

The New Zealand project used two methods for evaluating the impact of the teaching approaches. A pre and post-test was given to the students, and the results on the post-test were compared with results on a parallel test given to a comparison group consisting of a representative sample of the Year 9 population. Assessment of the students in the post-test showed significant improvements in algebraic strategies and knowledge compared to the pre-test. Furthermore, the measures of student outcomes displayed significantly higher values for students in the project compared to the comparison group. The improvement from pre to post-test was largely due to far fewer students in the post-test using the most primitive strategies (e.g., using guess and check to solve equations) and displaying very poor knowledge. At post-test there were, however, still relatively few students using the most sophisticated strategies (e.g., transforming equations by doing the same thing to both sides).

A number of items used in the New Zealand assessment were similar to items used in the Australian assessment. One question was:

> There are 4 classes at Waitati school and every class has the same number (n) of students in it. How many students are at the school altogether?

For these students and this context we did not see examples of pronumerals being used as labels or objects, but instead saw the error described by MacGregor and Stacey (1997) as "letter interpreted as a numerical value". In the pre-test 29% of students gave an arbitrary value to n and calculated

a value, but in the post-test this proportion dropped to 22% on a directly parallel question.

Another question was:

> I have n sheep on my farm and each one gave birth to twin lambs, but 5 lambs died in a snowstorm. How many lambs are left?

In the pre-test 24% of students gave an arbitrary value to n and calculated a value for this question also, and in the post-test this proportion dropped to 16% on a directly parallel question. For those students who gave an algebraic expression as an answer, there were no instances of inappropriate conjoining of terms in the pre-test, possibly because $2n - 5$ could not be conjoined easily, whereas 6% of students inappropriately conjoined $2n + 14$ as $16n$ or $14n^2$ in the post-test.

Implications for teaching, learning and assessment

Even though the Australian project's teaching was based firmly on a teaching resource (McMaster & Mitchelmore, 2009) and the teaching approaches in the New Zealand project were diverse and not pre-determined, there were strong similarities in what occurred in the experimental classrooms of each country. Both projects made extensive use of meaningful contexts and did not teach algebraic skills out of context. Because of the contexts, students were able to tackle problems in a variety of different ways, which encouraged discussion. This use of meaningful contexts is similar to approaches advocated in many other projects (Baroudi, 2006; Green, 2008, 2009; MacGregor & Stacey, 1997; Samson, 2012; Shield, 2008) and the students in our projects displayed clear evidence of enhanced learning of algebra compared to students in the comparison groups. The largest improvements were for the students with the least knowledge and most primitive strategies in the New Zealand project, while in the Australian project it was the students in the lowest ability class (Experimental Class 4) who performed better than those in the comparison group (Comparison Class 3). The teaching approaches therefore appeared to have had most impact on those students who struggle to make the transition from arithmetic to algebra.

The extensive use of contexts is consistent with promoting the learning of structural knowledge rather than just procedural knowledge. Both types of mathematical knowledge are needed for solving problems effectively when the problems are set in meaningful contexts. Teachers who have taught conventional algebra courses will be aware of just how difficult students find application problems, if they are given at the end of a unit of work which consisted of learning routine skills. Procedural knowledge by itself is of little use for solving contextual problems requiring interpretation of the mathematical structure. It is therefore fascinating to observe that the converse is not true. Students in the experimental classes were not disadvantaged by the lack of practice of routine skills. Although these classes had spent considerable time working with rich contexts the students still managed to learn the procedural knowledge without spending a lot of time on routine exercises.

A number of misconceptions about algebra identified by MacGregor and Stacey (1997) were apparent in our data, and were reduced in the experimental classes. The misconceptions displayed in different questions were, however, quite different. Even though questions appeared to be structurally

equivalent, the particular context appeared to make a big difference. These differences cannot be accounted for as differences between the two countries. In the New Zealand project, even though the frequency of misconceptions decreased from pre-test to post-test, the types of misconceptions changed with the particular context used. We should therefore be particularly cautious in trying to generalise from errors we observe students making.

This research has considerable significance for teachers of students who are struggling with introductory high school algebra. The lower ability students, in particular, needed to be engaged in working with meaningful contexts that allowed them to use their existing knowledge and skills to solve problems. The varieties of methods used by the students provided valuable opportunities for the teachers to lead discussions that gave meaning to algebra. Even though investigating contextual problems and taking part in discussions may seem to be time-consuming, the students did not appear to be disadvantaged in any way. We recommend that teachers should look for opportunities for using algebra everywhere. When algebraic tools are used throughout the year students are less likely to view algebra as irrelevant to their lives.

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